

Comment on "Convergence of macrostates under reproducible processes" [Phys. Lett. A 374: 3715-3717 (2010)]

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Abstract

J. Rau derived in [Phys. Lett. A 374: 3715-3717 (2010)] a monotonicity property of relative entropy and obtained the *superadditivity inequality* – a much stronger monotonicity – of relative entropy:

$$S(\rho_{AB}||\sigma_{AB}) \geq S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B),$$

where ρ_{AB} and σ_{AB} are bipartite states on $\mathcal{H}_A \otimes \mathcal{H}_B$. We provide a simple counterexample to show that the above inequality is not correct.

1 Introduction

Relative entropy are powerful tools in quantum information theory [1]. It has a monotonicity property under quantum channels [2]. In [3, 4], Petz studied (strong) superadditivity of relative entropy. J. Rau derived in [5] a monotonicity property of relative

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entropy by Jaynes' argument for the second law as a motivation, i.e., monotonicity of relative entropy under nonlinear coarse-graining. From which he derived a *superadditivity inequality* of relative entropy:

$$S(\rho_{AB}||\sigma_{AB}) \geq S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B), \quad (1)$$

where ρ_{AB} and σ_{AB} are bipartite states on tensor space $\mathcal{H}_A \otimes \mathcal{H}_B$, ρ_A, ρ_B, σ_A and σ_B are the reduced density matrices of ρ_{AB} and σ_{AB} respectively. The goal of this Letter is to provide a simple counterexample which violates the inequality Eq. (1).

Let $\mathcal{D}(\mathcal{H})$ denote the set of all the density matrices ρ on \mathcal{H} . The *von Neumann entropy* $S(\rho)$ of ρ is defined by

$$S(\rho) \stackrel{\text{def}}{=} -\text{Tr}(\rho \log \rho).$$

The *relative entropy* [1] of two mixed states ρ and σ is defined by

$$S(\rho||\sigma) \stackrel{\text{def}}{=} \begin{cases} \text{Tr}(\rho(\log \rho - \log \sigma)), & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}(\mathcal{H})$ be the set of all linear operators on \mathcal{H} and $\mathcal{T}(\mathcal{H})$ be the set of all linear maps from $\mathcal{L}(\mathcal{H})$ to itself. $\Lambda \in \mathcal{T}(\mathcal{H})$ is said to be a *completely positive linear map* if for each $k \in \mathbb{N}$,

$$\Lambda \otimes \mathbb{1}_{M_k(\mathbb{C})} : \mathcal{L}(\mathcal{H}) \otimes M_k(\mathbb{C}) \rightarrow \mathcal{L}(\mathcal{H}) \otimes M_k(\mathbb{C})$$

is positive, where $M_k(\mathbb{C})$ is the set of all $k \times k$ complex matrices. It follows from Choi's theorem [6] that every completely positive linear map Λ has a Kraus representation

$$\Lambda = \sum_{\mu} \text{Ad}_{M_{\mu}},$$

that is, for every $X \in \mathcal{L}(\mathcal{H})$, $\Lambda(X) = \sum_{\mu} M_{\mu} X M_{\mu}^{\dagger}$, where $\{M_{\mu}\} \subseteq \mathcal{L}(\mathcal{H})$, $\sum_{\mu=1}^K M_{\mu}^{\dagger} M_{\mu} = \mathbb{1}_{\mathcal{H}}$, M_{μ}^{\dagger} is the adjoint operator of M_{μ} . A *quantum channel* is just a trace-preserving completely positive linear map Φ .

2 A counterexample

Let $|\psi_X\rangle, |\phi_X\rangle \in \mathcal{H}_X$ such that $\langle \psi_X | \phi_X \rangle = 0$, where $X = A, B$. Set

$$\rho_{AB} = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$$

and

$$\sigma_{AB} = \lambda |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| + (1 - \lambda) |\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B|,$$

where $\lambda \in (0, 1)$. We have

$$S(\rho_{AB}||\sigma_{AB}) = S(\rho_A||\sigma_A) = S(\rho_B||\sigma_B) = h(\lambda),$$

where $h(\lambda) = -\lambda \log \lambda - (1 - \lambda) \log(1 - \lambda)$, which implies that

$$S(\rho_{AB}||\sigma_{AB}) < S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B).$$

This counterexample also suggests that J. Rau's result concerning the monotonicity of relative entropy under nonlinear coarse-graining [5] is not valid in general:

$$S(\mu_{f(\rho)}||\mu_{f(\sigma)}) \not\leq S(\rho||\sigma). \quad (2)$$

As the results have been cited for several times, e.g. in [7], the related results based on Eqs. (1) and (2) possibly need to be reconsidered.

3 Discussion

If the superadditivity inequality of relative entropy Eq. (1) were valid, one would have another equivalent form of this inequality.

Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. Given a quantum channel $\Phi \in \mathcal{T}(\mathcal{H}, \mathcal{K})$ with a Kraus representation:

$$\Phi = \sum_{\mu=1}^K \text{Ad}_{M_\mu},$$

where $M_\mu \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ for all μ such that $\sum_{\mu=1}^K M_\mu^\dagger M_\mu = \mathbb{1}_{\mathcal{H}}$. Taking a complex Hilbert space $\mathcal{H}_E = \mathbb{C}^K$ with an orthonormal basis $\{|\mu\rangle : \mu = 1, \dots, K\}$, and defining

$$V|\psi\rangle \stackrel{\text{def}}{=} \sum_{\mu} M_\mu |\psi\rangle \otimes |\mu\rangle, \quad \forall |\psi\rangle \in \mathcal{H}.$$

By the Stinespring representation of quantum channels, one has

$$\Phi(\rho) = \text{Tr}_E (V \rho V^\dagger).$$

The corresponding complementary channel is

$$\widehat{\Phi}(\rho) = \text{Tr}_{\mathcal{K}} (V \rho V^\dagger) = \sum_{\mu, \nu=1}^K \text{Tr} (M_\mu \rho M_\nu^\dagger) |\mu\rangle\langle\nu|.$$

Clearly, $V \in L(\mathcal{H}, \mathcal{K} \otimes \mathcal{H}_E)$ is a linear isometry so that $V\tau V^\dagger$ has, up to multiplicities of zero, the same eigenvalues as τ for all $\tau \in D(\mathcal{H})$. Then it follows that

$$\begin{aligned} S(\rho||\sigma) &= S(V\rho V^\dagger||V\sigma V^\dagger) \\ &\geq S\left(\text{Tr}_E(V\rho V^\dagger)||\text{Tr}_E(V\sigma V^\dagger)\right) + S\left(\text{Tr}_K(V\rho V^\dagger)||\text{Tr}_K(V\sigma V^\dagger)\right) \\ &= S(\Phi(\rho)||\Phi(\sigma)) + S(\hat{\Phi}(\rho)||\hat{\Phi}(\sigma)). \end{aligned}$$

Therefore

$$S(\rho||\sigma) \geq S(\Phi(\rho)||\Phi(\sigma)) + S(\hat{\Phi}(\rho)||\hat{\Phi}(\sigma)). \quad (3)$$

Now taking $\rho = \rho_{AB}$ and $\Phi(\rho_{AB}) = \text{Tr}_B(\rho_{AB})$, we have $\hat{\Phi}(\rho_{AB}) = W\rho_B W^\dagger$ for some linear isometry $W \in L(\mathcal{H}_B, \mathcal{H}_E)$. By employing Eq. (3), we have

$$\begin{aligned} S(\rho_{AB}||\sigma_{AB}) &\geq S(\Phi(\rho_{AB})||\Phi(\sigma_{AB})) + S(\hat{\Phi}(\rho_{AB})||\hat{\Phi}(\sigma_{AB})) \\ &= S(\rho_A||\sigma_A) + S(W\rho_B W^\dagger||W\sigma_B W^\dagger) \\ &= S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B). \end{aligned}$$

Since Eqs. (1) and (3) are not valid in general, a question naturally arises: what is the largest class of states such that Eq. (1) is satisfied? Or which channels can preserve the inequality Eq. (3)?

Remark 3.1. In [8], the authors proposed the following question: For given quantum channel $\Phi \in T(\mathcal{H}_A, \mathcal{H}_B)$ and states $\rho, \sigma \in D(\mathcal{H}_A)$, does there exist a quantum channel $\Psi \in T(\mathcal{H}_B, \mathcal{H}_A)$ with $\Psi \circ \Phi(\sigma) = \sigma$ and

$$S(\rho||\sigma) \geq S(\Phi(\rho)||\Phi(\sigma)) + S(\rho||\Psi \circ \Phi(\rho))? \quad (4)$$

The authors affirmatively answer this question in the classical case. The quantum case is still open.

Based on the above discussion, we come up with the following questions:

- (i) Can we have $\hat{\Phi}(\rho) = \hat{\Phi}(\sigma)$ if $S(\rho||\sigma) = S(\Phi(\rho)||\Phi(\sigma))$?
- (ii) What can be derived from $\hat{\Phi}(\rho) = \hat{\Phi}(\sigma)$?
- (iii) What can be derived from $S(\rho||\sigma) = S(\hat{\Phi}(\rho)||\hat{\Phi}(\sigma))$?

Remark 3.2. For (i), M. Hayashi gives the following negative answer in [9]. Let ρ, σ and Φ be as follows:

$$\rho = \sum_j \lambda_j(\rho) E_j, \quad \sigma = \sum_j \lambda_j(\sigma) E_j, \quad \Phi(X) = \sum_j E_j X E_j,$$

where each E_j is a projector such that $\sum_j E_j = \mathbb{1}$, which implies that $S(\rho||\sigma) = S(\Phi(\rho)||\Phi(\sigma))$ and

$$\hat{\Phi}(\rho) = \sum_j \lambda_j(\rho) |j\rangle\langle j|, \quad \hat{\Phi}(\sigma) = \sum_j \lambda_j(\sigma) |j\rangle\langle j|.$$

Now if $\lambda(\rho) \neq \lambda(\sigma)$, then $\hat{\Phi}(\rho) \neq \hat{\Phi}(\sigma)$.

We see that no matter how close ρ and σ are, Eq. (3) does not hold. Therefore we guess that Eq. (1) even does not hold when ρ_{AB} is close to σ_{AB} in some sense.

4 Weaker superadditivity inequality: a conjecture

Although the superadditivity inequality is not valid in general, we can propose some approach to its weaker version. In [10], we have obtained the following result:

Theorem 4.1. For given two quantum states $\rho, \sigma \in D(\mathcal{H}_d)$, where σ is invertible, it holds that

$$\min_{U \in U(\mathcal{H}_d)} S(U\rho U^\dagger || \sigma) = -S(\rho) - \sum_j \lambda_j^\downarrow(\rho) \log \lambda_j^\downarrow(\sigma), \quad (4.1)$$

$$\max_{U \in U(\mathcal{H}_d)} S(U\rho U^\dagger || \sigma) = -S(\rho) - \sum_j \lambda_j^\downarrow(\rho) \log \lambda_j^\uparrow(\sigma), \quad (4.2)$$

where $\lambda_j^\downarrow(\sigma)$ stands for the eigenvalues arranged in decreasing order and $\lambda_j^\uparrow(\sigma)$ stands for the eigenvalues arranged in increasing order. $U(\mathcal{H}_d)$ denotes the set of all unitary operators on \mathcal{H}_d .

Based on the above result, we propose the following *conjecture*: There exist three unitary operators $U_A \in U(\mathcal{H}_A)$, $U_B \in U(\mathcal{H}_B)$ and $U_{AB} \in U(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that

$$S(U_{AB}\rho_{AB}U_{AB}^\dagger || \sigma_{AB}) \geq S(U_A\rho_AU_A^\dagger || \sigma_A) + S(U_B\rho_BU_B^\dagger || \sigma_B). \quad (4.3)$$

where the reference state σ is required to be an invertible non-product state.

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